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On the sine-Gordon soliton mass at finite temperatures

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Abstract. There exist some criticisms of the Bethe ansatz formulation predicting a discontinuity for the sine-Gordon soliton mass at finite temperatures. The basic ideas and some key steps of the Bethe ansatz thermodynamics are examined and clarified along with associated criticisms. The validity of the Bethe ansatz theory is demonstrated and an experimental implication is discussed.

Over the last several years, a serious controversy has arisen over the Bethe ansatz (BA) formulation of the equilibrium statistical mechanics of the sine-Gordon (SG) model, particularly the soliton mass at finite temperatures. Chung and Chang [1] claimed that the finite-temperature soliton mass is discontinuous as a function of coupling constant within a BA theory, whereas Johnson and Fowler [2, 3] and Bullough [4] insist that the soliton mass cannot be discontinuous. The problem is not simply a terminological misconception, as is clear in a recent article by Fowler and Johnson (FJ) [3]. They argued that the soliton mass discontinuity arises from the existence of certain zero-binding-energy states, and therefore it is merely an artifact of the Chung and Chang BA thermodynamic formulation. Moreover, FJ and Timonen *et al* [5] proposed a definition of soliton mass which shows continuity at a series of coupling constants where our theory predicts a discontinuity. In this paper, I shall fully examine and clarify the BA thermodynamic formulation in its basic ideas and some key steps. It will be demonstrated that the Chung and Chang BA theory is free from errors, whereas the existing criticisms are inconclusive. Experimentally, I predict that something like the broken curve in figure 1 would be observed for the finite-temperature SG soliton mass as a function of the coupling constant.

Let us start with elementary excitations in the SG model; solitons, antisolitons and their bound states, i.e. breathers. As is well known, the SG model is equivalent to the massive Thirring model, with the identification: solitons → fermions, antisolitons → antifermions and breathers → fermion–antifermion bound states. On the other hand, the BA theory works on breathers, holes in the Dirac sea and long strings or Korepin (K-) excitations and needs to identify holes plus K-excitations in terms of fermions → solitons and antifermions → antisolitons. Concerning this identification, what one could show within a BA theory is that in the charge neutral sector, holes plus K-excitations are always identified as equal numbers of unbound solitons and antisolitons as far as energy and momentum are concerned. K-excitations are unphysical objects, i.e. they do not contribute to physical quantities such as energy, momentum and free energy [6]. One concludes from this that the energies, momenta and free energies carried by holes are exactly twice those of solitons in the charge neutral sector. However, in addition to this information, the BA thermodynamics formulation requires an explicit

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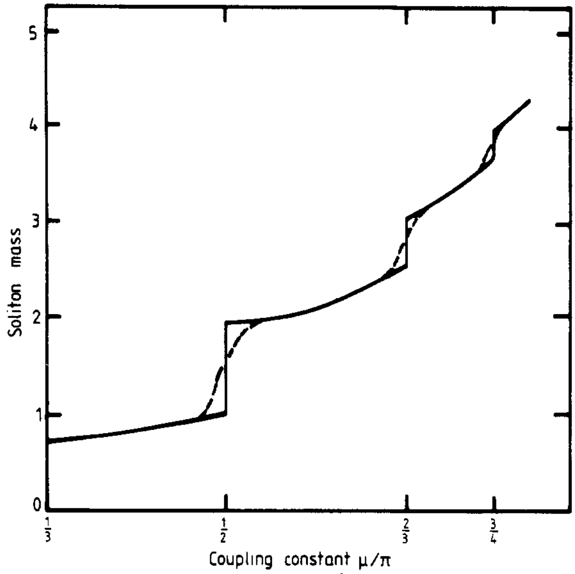

Figure 1. Soliton mass as a function of the coupling constant at $T=2$. The mass and temperature are measured in the unit of the zero-temperature free soliton mass. The full curve is the Bethe ansatz/factorised S-matrix method result, whereas something like the broken curve is expected to be observed experimentally.

form of the free energy contribution from solitons, which is beyond the ability of the BA theory. This is simply because the BA theory cannot describe a soliton-antisoliton backscattering without the help of the K-excitations, which means that the solition or antisoliton degree of freedom cannot be singled out in the BA theory [7]. It is in this point that the factorised S-matrix method developed by Chung *et al* [8] is superior to the BA method; unlike the latter, the former method allows us to write the quantisation of soliton momentum in the same form as that of hole momentum, which implies that the free energy carried by the soliton can be written in the same form as that carried by the hole. In this way, the crucial equations (2) and (3) below are justified.

Given temperature T and coupling constant μ, our goal is to find the thermally renormalised energy-momentum dispersion relations for soliton, antisoliton and breathers, their concentrations and respective contributions to the free energy. There is no particular role in the BA thermodynamics for the heat-bath†. Although the BA formulation applies a variational principle to the free energy, the effect of the heat bath is essentially the Boltzmann factor $\exp(-H/kT)$. That is, the BA formulation *exactly* describes the equilibrium statistical mechanics of the SG model. Thus the unusual discontinuity found in the soliton mass at finite temperatures, if true, is due to an unusual property of the SG model, i.e. a complete integrability. The point in question, then, is how well a completely integrable model can describe real physical systems. A well known example in this respect is the Kondo model, solved by Bethe ansatz [9], to describe a magnetic impurity in the sea of conduction electrons. The BA analysis predicts a specific heat and susceptibility in excellent agreement with experiment. Indeed, FJ pointed out this fact to insist that the BA thermodynamics is

† Our argument in [1] (pp 2891, 2892) about the heat bath is erroneous. It should be corrected as in the present paper.

not as fragile as it might seem. This might be true for macroscopic quantities like specific heat which are smooth functions of the coupling constant, as was rigorously proved by Araki [10] and explicitly shown by the BA analysis for the SG model, but is not quite so obvious for microscopic quantities like soliton mass.

Here we look at criticisms of the soliton mass discontinuity. Fowler and Johnson [3] insist that the BA results for soliton mass should be experimentally observed without modification (like specific heat in the Kondo problem), and therefore a discontinuity is unphysical. They propose a definition of soliton mass which shows continuity at the series of coupling constants in question. On the other hand, Bullough [4] insists that the identification of holes plus K-excitations in terms of solitons and antisolitons is impossible, and therefore the soliton mass should be defined such that it does not show an unphysical discontinuity. As is demonstrated in the above, the identification problem is positively resolved. In the rest of the paper, I will examine essential steps of the BA analysis concerning the soliton mass along with associated criticisms. In the end, I will reach the conclusion that the FJ argument is far from convincing, and their definition of soliton mass is neither logically consistent nor well grounded. As for the discontinuous jump in the soliton mass, as shown by a full curve in figure 1, it should be regarded as *a mathematical singularity of the idealistic, i.e. completely integrable,* SG *model.* In reality, small non-integrable perturbations always exist, and such a singularity would be more or less relaxed to something like a broken curve in the figure. Although the physics is different, similar situations are often encountered; a familiar example is superconductivity. In this second-order phase transition, a specific heat should ideally have a discontinuous jump at the critical point, but in reality the jump is always rounded off due to inhomogeneity etc.

We now take a close look at the BA analysis. The soliton mass discontinuity in question occurs as the coupling parameter passes through the values $\mu=\pi n/(n+1)$, where n is an integer. In the intervals $\pi(n-1)/n<\mu\leqslant\pi n/(n+1)$ we have $n-1$ different breathers as well as solitons and antisolitons, and all the quantities are continuous in these intervals. To be more specific and for simplicity, consider $\mu=\pi/2$ and $\pi/2<\mu\leqslant 2\pi/3$. The former case describes non-interacting solitons and antisolitons. In the latter case, we have one type of breather in addition to solitons and antisolitons. The BA analysis of the latter case is as follows. Consider $\mu=\pi(1+m)/(1+2m)$ (integer $m\geqslant 1$) and let ε^{b}, ε^{h} and $\varepsilon_j^{\mathrm{k}}$ ($j=1,2,\ldots,m-1$), respectively, be renormalised dispersion curves of the breather, the hole and the K-excitations. The coupled integral equations for these quantities have the following structure [1]:

$$\varepsilon^{\mathrm{b}}=E^{\mathrm{b}}+F^{\mathrm{b}}(\varepsilon^{\mathrm{b}},\varepsilon^{\mathrm{h}}) \tag{1a}$$

$$\varepsilon^{\mathrm{h}}=E^{\mathrm{h}}+F^{\mathrm{h}}(\varepsilon^{\mathrm{b}},\varepsilon^{\mathrm{h}},\{\varepsilon^{\mathrm{k}}\}) \tag{1b}$$

$$\varepsilon_j^{\mathrm{k}}=F_j^{\mathrm{k}}(\varepsilon^{\mathrm{h}},\{\varepsilon^{\mathrm{k}}\}) \tag{1c}$$

where E^{b} and E^{h} are excitation spectra for breathers and holes at $T=0$ and F^{b}, F^{h} and F_j^{k} denote certain functionals. Explicit forms are given in [1], but are not necessary here. As is discussed in the above, the soliton contribution to the local (in rapidity space) free energy is written as

$$-\frac{\gamma T}{2\pi}E^{\mathrm{h}}\ln[1+\exp(-\varepsilon^{\mathrm{s}}/T)] \tag{2}$$

and the hole free energy should be twice the soliton free energy, which with the help of (2) implies that

$$\ln[1+\exp(-\varepsilon^h/T)] = 2\ln[1+\exp(-\varepsilon^s/T)] \tag{3}$$

where γ is a field theoretic renormalisation factor and ε^s is the dispersion curve of the SG soliton. Solving the integral equations (1) and (3), we obtain ε^s which, for vanishing momentum, represents the soliton mass at finite temperatures. Such an analysis was done before for the entire attractive regime [1], and the full curve in figure 1 shows the soliton mass as a function of coupling constant.

What then, is controversial in the above BA analysis? It is the physical meaning of the ε functions. While the ε functions are proved to be the excitation spectra in [1], it is argued by FJ that there are some special cases where the ε function does not correspond directly to a physical excitation curve at finite temperature. Details will be given below, but my point against their criticism is that the BA analysis including the above mentioned proof *equally applies* over the entire interval $\pi(n-1)/n<\mu\leqslant \pi n/(n+1)$.

The physical origin of the discontinuous jump is clear. Each time μ passes through $\pi n/(n+1)$, $n=1,2,\ldots$, from above, the highest-energy breather dissociates into a soliton–antisoliton pair. Since the free energy is a continuous function of μ, a sudden disappearance of the free energy due to the breather dissociation should be compensated by a sudden increase of the free energy due to solitons and antisolitons, which due to (2) results in a sudden decrease of the soliton mass. Now, the FJ argument against the discontinuity is that if one looks at $\mu=\pi/2$, for example, it is simply non-interacting solitons and antisolitons and the discontinuity arises when one considers a zero-binding-energy breather as an independent excitation. If we looked only at the point $\mu=\pi/2$, nobody would try to rearrange free solitons and antisolitons into solitons, antisolitons and zero-binding-energy breathers. One should rather consider the coupling interval $\mu=\pi/2+\delta$; $0<\delta\leqslant\pi/6$ and bear in mind that this spuriously looking zero-binding-energy breather is obtained in the limit $\delta\to 0$. In this coupling segment, we have soliton, antisoliton and one type of breather with in general a finite binding energy relative to an unbound soliton–antisoliton pair, and all the quantities are analytic. There is no particular difference in the description among $\delta=\pi/6$, 0.1, 0.001 and 0. Due to the complete integrability of the SG model, the breather is always a well defined excitation no matter how small the binding energy. Fowler and Johnson also gave a definition of soliton mass (cf equation (6) in [3]; note that our equation (3) is *not* a definition but is derived) just for the point $\mu=\pi/2+0$. With no plausibility argument given (it is *not* forced out by a free-energy consideration or the like), it is an arbitrary definition only for the sake of continuity in the soliton mass. In short, they essentially consider only the point $\mu=\pi/2$, not the proper connection between the point $\mu=\pi/2$ and the segment $\pi/2<\mu\leqslant 2\pi/3$.

Finally, consider the free-phonon limit $\mu\to\pi-0$. At exactly $\mu=\pi$, the system represents non-interacting phonons, whereas at $\mu\to\pi-0$, the system is described by an infinite number of breathers all of which look like zero-binding-energy bound states of phonons. If one looks only at this limit, there is no need for breathers. However, if one wishes to describe a general coupling regime exactly, the necessity of breathers is unquestioned. One should, rather, take a viewpoint that the BA theory which exactly describes the thermodynamics of the SG model for general coupling constants indeed contains the free-phonon theory as a special limit. Here Johnson and Fowler [2] develop a similar argument as in the free-fermion case, and make the criticism that while the phonon mass is temperature independent, the lowest breather mass is strongly temperature dependent. From this, they conclude again that the ε function in the BA theory does not correspond directly to a physical excitation curve at finite temperatures. However, their conclusion is based on an erroneous identification between the linear phonon and the lowest-energy breather. They are not the same object; the phonon is a boson, but all the breathers are essentially fermions.

To conclude, the validity of the Chung and Chang Bethe ansatz thermodynamic theory predicting a discontinuous soliton mass at finite temperatures is demonstrated. An experimental observation of the variation of the finite-temperature soliton mass with coupling constant in a real physical system is eagerly awaited.

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